Finite Element Stress Analysis: Theory; failure criteria; Example Application

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What is a process model?

- Makes quantitative predictions about process

Example Model:

How long is a planet’s orbit (years)

Kepler’s 3rd Law of Planetary Motion, 1619

Distance from Sun
(in Earth Orbital Radii) \[ P = R^{3/2} \]

Orbit Period
(in Earth years)

Mars:

\[ P = (1.52)^{3/2} \]
\[ = 1.88 \text{ years} \]

(matches observation)

Pluto:

\[ P = (39.44)^{3/2} \]
\[ = 248 \text{ years} \]

(discovered in 1930)

Steps in Modeling - summary

1. Define the real-world problem and model (including constants)
2. Calibrate model to match known data (use dimensionless numbers) - Earth
3. Validate the model - Mars
   - compare with known solution
4. Use the model
   - learn something new! - Pluto
5. Extend the model further (needs to be more fundamental) - other solar system
What is model implementation?

Inducing beneficial changes to process operation


Literature Models

Graph containing results which can be implemented

Something you care about

Something you can do something about

Why Model?

- increase fundamental understanding
- technology transfer
- design of experiments
- evaluation of alternative designs
- process optimization
- extension and evaluation of plant results
- extending lab measurements to quantify properties
- assist in scale-up
- online process control

If a clear reason to develop a model cannot be found, then it should not be developed!

Find FEM Equations

1) Review Governing Eqns.-3D Elasticity

- Unknown field variables to solve for:
  - 3 displacements: $u, v, w$
  - 6 strains: $\varepsilon_x, \varepsilon_y, \varepsilon_z, \gamma_{xy}, \gamma_{yz}, \gamma_{zx}$
  - 6 stresses: $\sigma_x, \sigma_y, \sigma_z, \tau_{xy}, \tau_{yz}, \tau_{zx}$

- Equations to solve:
  - 3 equilibrium equations: $[A]^T \{\sigma\} = \{F\}$
  - 6 constitutive equations: $\{\sigma\} = [D] \{\varepsilon\}$
  - 6 strain-displacement eqs.: $\{\varepsilon\} = [A] \{u\}$
Find FEM Eqs.: Galerkin’s Method

- Substitute:

\[
\begin{aligned}
\{\sigma\} &= [D]\{\epsilon\} = [D][A]\begin{bmatrix} u \\ v \\ w \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} u \\ v \\ w \end{bmatrix} = [N]\{d\}
\end{aligned}
\]

\[
\int \left( [A][N]^T \right) \left( [D][A] \right) dV \{d\} = \int \left( [N]^T \right) \{\sigma\} dS + \int \left( [N]^T \right) \{F\} dV
\]

\[
\int \left( \begin{bmatrix} B \\ D \end{bmatrix} \right)^T \left( \begin{bmatrix} B \\ D \end{bmatrix} \right) dV \{d\} = \int \left( [N]^T \right) \{F\} dV + \int \left( [N]^T \right) \{\sigma\} dS
\]

- \( [K] \) units

\[
\begin{bmatrix} \frac{1}{l} \\ \frac{N}{m} \\ \frac{1}{l} \end{bmatrix} m = \frac{N}{m} \quad \text{and} \quad \left( \frac{N}{m^2} \right) m = N
\]

Global Equation System

\[
[K]{d} = \{F\}_{bf} + \{F\}_{\Phi} + \{F\}_{\epsilon_i} + \{P\}
\]

- Force Vectors

\[
\{F\}_{bf} = \int [N]^T \{F\} dV \quad \text{body forces} \quad \left( \frac{N}{m} \right) (m') = N
\]

\[
\{F\}_{\Phi} = \int [N]^T \{\Phi\} dS \quad \text{surface tractions} \quad \left( \frac{N}{m^2} \right) (m') = N
\]

\[
\{F\}_{\epsilon_i} = \int [B]^T [D]\{\epsilon_0\} dV \quad \text{Initial strain forces} \quad \sigma = [D]\left( \left\{\epsilon\right\} - \left\{\epsilon_0\right\}\right)
\]

\[
\{P\} = \text{point loads (N)} \quad \left\{\epsilon_0\right\} = \text{"initial strain"}; \quad \text{plastic, creep, thermal, etc.}
\]
Stress Boundary Conditions

- Specify 1 of the following on each portion of the domain boundary for each DOF (2 DOF for 2-D problems and 3 for 3-D):

  On $S_1$, Specify Displacement(s)
  \[ u_N = \overline{u}_N(x, y, z) \]
  \[ u_T = \overline{u}_T(x, y, z) \]

  On $S_2$, Specify Surface Traction(s)
  \[-E \frac{\partial u_N}{\partial n} = \Phi_{\text{pressure traction}} \]
  \[-E \frac{\partial u_T}{\partial n} = \Phi_{\text{shear traction}} \]

  On chosen nodes: Point Load(s)

\{F_\Phi\} Surface Traction

Force Vector on Side of a CST

- Apply traction $\Phi (N/m^2)$ on side 1-2

example
Stress Analysis Output

- 6 strain components (isotropic) 
  \( \varepsilon_x, \varepsilon_y, \varepsilon_z \) (normal) \( \gamma_{xy}, \gamma_{yz}, \gamma_{zx} \) (shear)

- 6 stress components (isotropic) 
  \( \sigma_x, \sigma_y, \sigma_z \) (normal) \( \tau_{xy}, \tau_{yz}, \tau_{zx} \) (shear)

- Principle stresses, \( \sigma_1, \sigma_2, \sigma_3 \)
- Von Mises Effective stress,
- Stress intensity, SI

\[
\{ \varepsilon \} = [B] \begin{bmatrix} u \\ v \\ w \end{bmatrix}
\]

\[
\{ \sigma \} = [D] \{ \varepsilon \}
\]

Principle Stresses, \( \sigma_1, \sigma_2, \sigma_3 \)
(S1, S2, S3 in ANSYS)

- Max principle tensile stress, \( \sigma_1 = \max(\sigma_1, \sigma_2, \sigma_3) \)
- Max princ. compressive stress \( \sigma_3 = \min(\sigma_1, \sigma_2, \sigma_3) \)
- Find \( (\sigma_1, \sigma_2, \sigma_3) \) from 3 roots \( \sigma \) of:

\[
det \begin{vmatrix}
\sigma - \sigma & \tau_{xy} & \tau_{zx} \\
\tau_{xy} & \sigma - \sigma & \tau_{yz} \\
\tau_{zx} & \tau_{yz} & \sigma - \sigma \\
\end{vmatrix} = 0
\]

- Coordinate transformation to eliminate shear
Principal Stress Meaning

- Rotate coords until shear = 0

Stress Invariants, $I_1, I_2, I_3$

$$I_1 = \sigma_x + \sigma_y + \sigma_z$$

$$I_2 = \frac{1}{2} \left( \sigma_x^2 + \sigma_y^2 + \sigma_z^2 + 2\tau_{xy}^2 + 2\tau_{yz}^2 + 2\tau_{zx}^2 - I_1^2 \right)$$

$$I_3 = \text{det} \begin{vmatrix} \sigma_x & \tau_{xy} & \tau_{zx} \\ \tau_{xy} & \sigma_y & \tau_{yz} \\ \tau_{zx} & \tau_{yz} & \sigma_z \end{vmatrix}$$

- Use to find principal stresses from 3 roots of:

$$\sigma^3 - I_1\sigma^2 - I_2\sigma - I_3 = 0$$
Stress Intensity, SI

\( \text{SI} = \max \left( |\sigma_1 - \sigma_2|, |\sigma_2 - \sigma_3|, |\sigma_3 - \sigma_1| \right) \)

Von Mises Effective Stress, \( \overline{\sigma} \)

\[ \overline{\sigma} = \frac{1}{\sqrt{2}} \sqrt{\left(\sigma_x - \sigma_y\right)^2 + \left(\sigma_y - \sigma_z\right)^2 + \left(\sigma_z - \sigma_x\right)^2 + 6\left(\tau_{xy}^2 + \tau_{yz}^2 + \tau_{zx}^2\right)} \]

\[ = \frac{1}{\sqrt{2}} \sqrt{S_x^2 + S_y^2 + S_z^2} \quad \text{where} \quad S_j = \frac{1}{3}(2\sigma_j - \sigma_2 - \sigma_3) \]
Ductile Failure

Uniaxial loads

\( \sigma \) (yield surface - ductile fracture)

\( \sigma_{\text{yield}} \)

UTS

Uniaxial loads

\( \sigma_1 \)

\( \sigma_2 \)

Bi-axial loads

Von Mises Yield Criterion

Yield surface (2-D)

Also applies to polymers

\( \sigma_{\text{UTS}} \)

\( \sigma_{\text{YS}} \)

\( |\sigma_{\text{max}}\| = 1.5 \)

\( \epsilon_{\text{UTS}} \)

\( \epsilon_{\text{YS}} \)

\( \epsilon_{\text{max}} \)

PVC

PC

PS

PMMA

Figure 7.12 Biaxial yield data for various polymers compared to a modified octahedral shear stress theory. (After Raghava, Caddell, and Yeh [Raghava 73]; used with permission.)
Yield surface (3D)

For analysis of complex, 3-D, stress states

FIGURE 1.5. Geometrical representation of yield criteria in the principal stress space.

Brittle Failure

Results Analysis

• Brittle failure: \( \sigma_l > \sigma_{\text{max-B}} \)

\[ \sigma_{\text{max-B}} > \frac{K_f}{Y\sqrt{\pi a}} \]

• Ductile failure: \( \bar{\sigma} > \sigma_{\text{max-D}} \)

\[ \sigma_{\text{max-D}} = \sigma_{\text{Yield}} \]

Example:

Tri-axial Stress Condition

\( \sigma_z = 60 \)

\( \sigma_l = 60 \text{ MPa} \)

\( \sigma_3 = 60 \)

\( \sigma_{\text{max-D}} = 40 \text{ MPa} \)

\( \sigma_{\text{max-B}} = 55 \text{ MPa} \)

\[ \bar{\sigma} = \frac{1}{\sqrt{2}} \sqrt{(60-60)^2 + (60-60)^2 + (60-60)^2} \]

\( = 0 \) \( \Rightarrow \) Ductile Failure Impossible

But BRITTLE failure is likely
Combined Failure Criteria:
Yielding and Cracking

- **Ductile Failure**
  \[ \sigma > \frac{K_{lc}}{Y \sqrt{\pi a}} \]
  \[ \bar{\sigma} > \sigma_{Yield} \]

- **Brittle Failure**
  - Failure controlled by large flaws
  \[ \sigma_1 > \sigma_{max-B} \]

- **No Failure**
  - Failure controlled by yield strength
  \[ \sigma > \sigma_{max-D} \]

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Steps in Finite Element Analysis

1. Define the real-world problem
2. Transform to a mathematical problem
   - governing PDE(s), BCs, and domain
3. Solve using FEM
4. Check
   - analytical solutions and hand calculations
   - experiments
5. Parametric studies
6. Evaluation - recommendations